Problem Set on Matching Model of the Labor Market, Wage Functions, and Unemployment Fluctuations

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Problem 1

Consider a matching model with a labor force of size H. The matching function is Cobb-Douglas:

$$m(U, V) = \omega \cdot U^{\eta} \cdot V^{1-\eta},$$

where U is the number of unemployed workers, V is the number of vacant jobs, and $\eta \in (0, 1)$ is the matching elasticity. All workers are paid at a minimum wage $w > 0$. Firms have a production function

$$y(N) = a \cdot N^{\alpha},$$

where a governs labor productivity, N denotes the number of producers in the firm, and $\alpha \in (0, 1)$ indicates diminishing marginal returns to labor. Firms incur a recruiting cost of $r > 0$ recruiters per vacancy and face a job-destruction rate $s > 0$. The labor market tightness is $\theta = V/U$.

A) Compute the job-finding rate $f(\theta)$ and vacancy-filling rate $q(\theta)$. Assuming that labor-market flows are balanced, compute the recruiter-producer ratio $\tau(\theta)$.

B) Assuming that labor-market flows are balanced, compute labor supply $L^s(\theta, H)$.

C) Firms choose employment to maximize flow profits:

$$y(N) - [1 + \tau(\theta)] \cdot w \cdot N.$$

Compute the labor demand $L^d(\theta, a, w)$ by solving this maximization problem.

D) Give the equation that determines tightness in the model. Explain the origin of this equation.

E) Using the equation that you have just obtained, compute the elasticity of tightness with respect to the minimum wage, ϵ^{θ}_{w}. Is the elasticity positive or negative? Discuss your finding in light of the empirical literature on the minimum wage.

F) Now assume that productivity depends on the wage: $a(w) = \mu \cdot w^{\beta}$ with $\mu > 0$ and $\beta > 0$. Give possible reasons why $\beta > 0$. Recompute the elasticity of tightness with respect to the minimum wage under this new assumption. How does your answer compare with the answer you gave above?

Problem 2

Consider a large firm with $L(t)$ workers. There are two types of workers: $N(t)$ producers and $R(t)$ recruiters. All workers are paid a wage $w > 0$. The firm's production function is

$$y(t) = a \cdot N(t)^{\alpha},$$

where a governs labor productivity and $\alpha \in (0, 1)$ indicates diminishing marginal returns to labor. The firm face a job-separation rate $s > 0$, so it must post $V(t)$ vacancies to replace the workers who are leaving. Each vacancy requires the attention of $r > 0$ recruiters and is filled at a rate $q > 0$. The parameters satisfy $s \times r < q$. The firm discounts future profits at rate $\delta > 0$. The firm maximizes the discounted sum of future profits, taking the initial number of workers $L(0)$ as given.

A) Formulate the firm's problem using the number of producers $N(t)$ as control variable and the total number of employees $L(t)$ as state variable. Write down the associated current-value Hamiltonian.

B) Write down the optimality conditions for the firm's problem, and use them to derive the differential equation governing the optimal number of producers. The equation should be a first-order nonlinear differential equation involving $\dot{N}(t)$, $N(t)$, and parameters.

C) Compute the critical point of the dynamical system composed of the differential equations governing the evolution of $L(t)$ and $N(t)$ over time. How does the critical point relate to the labor demand computed in lecture? Under which conditions do they overlap?

D) Plot the phase diagram describing the evolution of employment in the firm. (The phase diagram should have $L(t)$ on the x-axis and $N(t)$ on the y-axis.) Does the phase diagram represent a sink, a source, or a saddle? Display the trajectory of employment for a given initial condition.

Problem 3

Consider a matching model with a labor force of size H. The matching function is Cobb-Douglas: $m(U, V) = \omega \cdot U^{\eta} \cdot V^{1-\eta}$, where U is the number of unemployed workers, V is the number of vacant jobs, and $\eta \in (0, 1)$ is the matching elasticity. All workers are paid at a minimum wage $w > 0$. Firms have a production function $y(N) = a \cdot N^{\alpha}$, where a governs labor productivity, N denotes the number of producers in the firm, and $\alpha \in (0, 1)$ indicates diminishing marginal returns to labor. Firms incur a recruiting cost of $r > 0$ recruiters per vacancy and face a job-destruction rate $s > 0$. The labor market tightness is $\theta = V/U$.

A) Compute the job-finding rate $f(\theta)$ and vacancy-filling rate $q(\theta)$. Assuming that labor-market flows are balanced, compute the recruiter-producer ratio $\tau(\theta)$. Compute the elasticities of f, q, and $1 + \tau$ with respect to θ.

B) Assuming that labor-market flows are balanced, compute labor supply $L^s(\theta, H)$. Compute the elasticities of L^s with respect to θ and H.

C) Firms choose employment to maximize flow profits: $y(N) - [1 + \tau(\theta)] \cdot w \cdot N$. Compute the labor demand $L^d(\theta)$ by solving this maximization problem. Compute the elasticity of L^d with respect to θ.

D) Characterize tightness $\theta(H)$ and unemployment rate $u(H)$ in the model. Illustrate with a diagram how $\theta(H)$ and $u(H)$ are determined.

E) Denote the elasticities of tightness $\theta(H)$ and unemployment rate $u(H)$ with respect to H as ϵ_H^{θ} and ϵ_H^{u}. Compute ϵ_H^{θ} and ϵ_H^{u}. Interpret the signs of these elasticities.

F) What is the sign of the derivative $d|\epsilon_H^{\theta}|/da$? (You do not need to compute the derivative: just find its sign.) There are times when people are strongly opposed to immigration, and other times when immigration is a nonissue. Based on the sign of the derivatives, when is it likely that opposition to immigration is particularly strong? As far as you know, is this prediction supported by historical evidence?